Options

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What's an Option's True Worth?

- We know the payoff at the expiration.
- We would like to know the value at inception and at any intermediate time t' between t and T. The market price should equal the value of the option.
- Let us denote an arbitrary option by O. We must have that O(t, T; K) at time t' has the same value as O(t', T; K). Why?
  Options have no "memory," i.e. the past (time < t) does not influence their value at time = t.
- We will thus only consider the value of O(t,T;K) at time t.

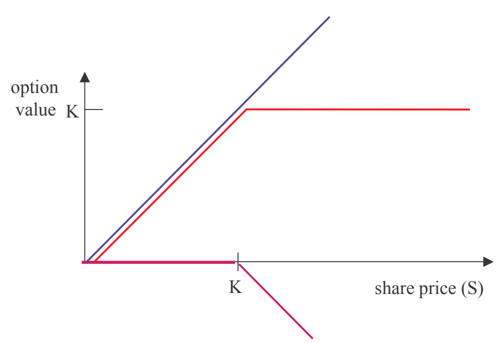
Call Inequalities

- **S(t) = 0 => C(t) = 0**
  Stock price = present value of future dividend payments. No cash flows, no value.
- **S(t) >= C(t)**
  If not true, we have arbitrage.
  Remember, an arbitrage is an opportunity to make money "for free." Know one? I am interested!

Call Inequalities (2)

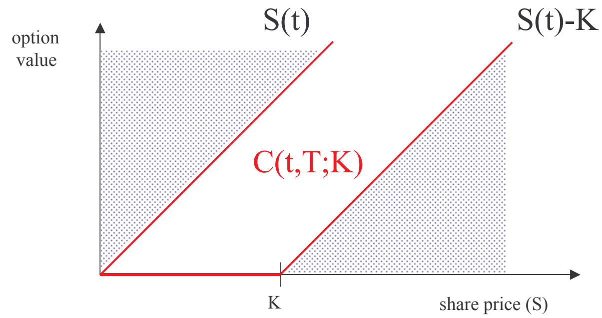
Assume S(t) < C(t). Buy stock, sell call at t.

	t	T
Cash flow	C(t)-S(t) > 0	S(T), if S(T) < K K, if S(T) >= K



Call Inequalities (3)

- **C(t) >= max(0, S(t) – K)**
  If we exercise at t, we get S(t) – K for sure.
  If we delay, we might get even more.
  On the other hand, we have limited liability.



Call Inequalities (4)

•C(T)=c(T)=max(0, S(T)-K)

•c(t)>=max(0, S(t)-K*p(t, T))

	t	T
Buy EU call	-c(t)	0, if S(T) < K S(T)-K, if S(T)>=K
Buy S and borrow K*p(t, T) at t, liquidate at T	-S(t)+K*p(t,T)	S(T) –K

Top strategy is better; it never pays less, and sometimes pays more. It must be that c(t)>=S(t)-K*p(t, T). But c(t) >= 0 (ltd. liability!)

Call Inequalities (4)

- **C(t) >= c(t)**
  An American call can always be used in lieu of a European call; it can not be worth less.
- If (a) no dividends, and (b) positive interest rates, then **C(t) = c(t)**.

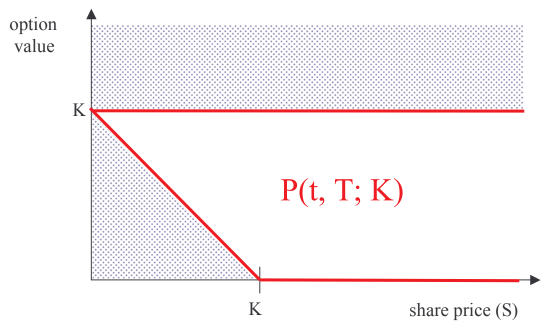
  An American call should never be exercised before expiration?! Yes (but check assumptions).

No Early Exercise of Am. Calls

- C(t) >= c(t) >= max(0, S(t)-K*p(t, T))
- C(t) >= S(t) – K*p(t,T)
- If exercised at t, C(t) = S(t) – K
- But t<T => p(t,T) < 1 (why?), hence
  C(t) >= S(t) – K*p(t,T) > **S(t) – K**.
- Hence C(t | **not** exercised) > C(t | exercised)
- Can this be right?

Put Inequalities

- **S(t) = 0 => P(t) = K**
- **K >= P(t)**
- **P(t) >= max(0, K – S(t)) (why?)**



Put Inequalities (2)

- Assume no dividends.
  p(t)>=Max(0, K*p(t,T)-S(t))

	t	T
Buy EU put	-p(t)	0, if S(T) > K K-S(T), if S(T)<=K
Agree to sell S at T for K (sell S fwd at T for K)	S(t)-K*p(t,T)	K-S(T)

Top strategy is better, it never pays less, and sometimes pays more. It must be that p(t)>=K*p(t, T)- S(t). But p(t) >= 0 (ltd. liability!)

This is what it costs today for somebody to enter into such an agreement.
At t, we buy S and borrow K*p(t,T) dollars. Then we can repay the loan, which grows to K dollars by time T, and we can still deliver the stock on our contract. Net profit? Zero! Is this reasonable?

Put Inequalities (3)

- **P(t) >= p(t)**
  An American option can be used in place of an European option; its value can not be less.

European Put-Call Parity at t

- c(T,T;K) - p(T,T;K) = S(T)-K (see above)
- c(t,T;K) - p(t,T;K) = S(t)-K*p(t,T)

"price of a default-free dollar"

Proof is obvious; set up two portfolios:
(a) Buy p(t, T; K), sell c(t,T;K);
(b) Buy S(t), borrow K units of the p(t,T) bond.
At the end, liquidate portfolios.
Payoff at T is identical.
Thus prices of the two portfolios must be identical.

American Put-Call Parity at t

- **$S(t)-K*p(t,T) >= C(t,T;K)-P(t,T;K) >= S(t)-K$**
- Remember: $c(t,T;K) - p(t,T;K) = S(t)-K*p(t,T)$
- For American instruments we do not have equality, but an interval of length $K*(1-p(t,T))$.
- Proof (part 1):
  $P(t) >= p(t) = c(t)-S(t)+K*p(t,T)$
  $P(t) >= C(t)-S(t)+K*p(t,T)$ (b/c $c(t)=C(t)$)
  $S(t)-K*p(t,T) >= C(t)-P(t)$

American Put-Call Parity at t (2)

- Proof (part 2):
  Goal: **$C(t,T;K)-P(t,T;K)-S(t)+K >= 0$**
  Create portfolio: buy call, sell put, borrow and sell S at t with the intent to buy back at T, or whenever the put is exercised; hold K in cash.
  We will **not** exercise the call **early**.
  (a) What happens if the put is exercised at t'?
  Note: Exercise implies $S(t') < K$.
  Payoff: $C(t')-\underbrace{[K-S(t')]}_{\text{payoff of put that we sold}}-\underbrace{S(t')}_{\text{cost of buying back stock}}+K=C(t') >= 0$

American Put-Call Parity at t (3)

- (b) What happens if the put is not exercised before T?

	S(T)<K	S(T)>=K
Call	0	S(T)-K
Put	-[K-S(T)]	0
Stock	-S(T)	-S(T)
Cash	K	K
Total	**0**	**0**

- Since the cash flow on this portfolio is non-negative under all scenarios, its value must be non-negative as well. This proves our relation.